

NORTH-HOLLAND MATHEMATICS STUDIES 177
(Continuation of the Notas de Matemática)

Editor: Leopoldo NACHBIN

*Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brazil
and
University of Rochester
New York, U.S.A.*



GROUP REPRESENTATIONS

Volume 2

Gregory KARPILOVSKY

*Department of Mathematics
California State University
Chico, CA, U.S.A.*



1993

NORTH-HOLLAND – AMSTERDAM • LONDON • NEW YORK • TOKYO

ELSEVIER SCIENCE PUBLISHERS B.V.
Sara Burgerhartstraat 25
P.O. Box 211, 1000 AE Amsterdam, The Netherlands

ISBN: 0 444 88726 1
(Volume 1 (Part A and B): ISBN 0 444 88632 X)

© 1993 ELSEVIER SCIENCE PUBLISHERS B.V. All rights reserved.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without the prior written permission of the publisher, Elsevier Science Publishers B.V., Copyright & Permissions Department, P.O. Box 521, 1000 AM Amsterdam, The Netherlands.

Special regulations for readers in the U.S.A. – This publication has been registered with the Copyright Clearance Center Inc. (CCC), Salem, Massachusetts. Information can be obtained from the CCC about conditions under which photocopies of parts of this publication may be made in the U.S.A. All other copyright questions, including photocopying outside of the U.S.A., should be referred to the publisher.

No responsibility is assumed by the publisher for any injury and/or damage to persons or property as a matter of products liability, negligence or otherwise, or from any use or operation of any methods, products, instructions or ideas contained in the material herein.

This book is printed on acid-free paper.

Printed in The Netherlands

For Helen

who helped me most

This page intentionally left blank

Preface

The present book is the second volume of a multi-volume treatise on group representations. The principal object of the volume is to provide, in a self-contained manner, a comprehensive coverage of projective representations and the Schur multiplier. Some further topics pertaining to projective representations will be covered in the next volume.

The book is a compilation and synthesis of a large body of work, done over several decades. It is aimed to give a clear and lucid exposition of the span of the field. The pace and style is appropriate for a graduate text, and could be used as an entry into the field as well as a basic reference. I hope that the volume will not only serve the specialist as a valuable reference, but will also provide ready access to the key ideas of the theory to the interested nonspecialist and to the student.

The bibliography is extensive, and it leads the reader to various references for detailed discussions on the main topics as well as on related subjects. A word about notation. As is customary, Theorem 4.3.2 denotes the second result in Section 3 of Chapter 4; however, for simplicity all references to this result within Chapter 4 itself, are designated as Theorem 3.2. The reader who wishes to obtain a more detailed summary account of the contents of this book can have it by reading through the brief introductions with which I begin each chapter.

I would like to express my gratitude to my wife for the tremendous help and encouragement she has given me in the preparation of this book. For answering specific queries on topics contained in the text I am indebted to R. Griess, D. Holt, J.-P. Serre, R. Steinberg, R. Swan, C. Thomas and T. Yoshida.

This page intentionally left blank

Contents

Preface	vii
Part I Projective Representations I	1
1. Second Cohomology Groups	3
1.1. Review of basic facts	4
1.2. Pairings, direct products and inflation maps	18
1.3. Abelian groups	28
1.4. A decomposition theorem	33
1.5. Cohomology groups over field coefficients	43
1.6. Semidirect products	48
1.7. An extension of the Hochschild-Serre exact sequence	58
2. Twisted Group Algebras	65
2.1. Equivalence	66
2.2. Commutativity	69
2.3. Semisimplicity, separability, symmetry and locality	72
2.4. Relations with group algebras	78
2.5. Natural homomorphisms	87
2.6. The center	93
2.7. Support of central idempotents	99

3. Introduction to Projective Representations	105
3.1. Definitions	105
3.2. Linear and projective equivalence	109
3.3. Projective representations and twisted group algebras	111
3.4. Contragredient representations	114
3.5. Inner tensor products	119
3.6. Outer tensor products	121
3.7. Projective representations of direct products	124
3.8. Induced representations	125
3.9. Frobenius reciprocity	127
3.10. Clifford's theorem	128
3.11. An application	130
4. Covering Groups	133
4.1. α -Covering groups	134
4.2. Covering groups	138
4.3. An application : faithful irreducible representations	145
4.4. Counting covering groups	147
4.5. Cohomology of covering groups	151
5. Degrees of Irreducible Representations	155
5.1. A preliminary result	156
5.2. Two theorems of Mackey	157
5.3. Degrees of irreducible representations	164
5.4. Applications : extensions of modules	170
6. Counting Irreducible Representations	177
6.1. Algebraically closed fields	177
6.2. Auxiliary results	179
6.3. Arbitrary fields	187
6.4. Counting projectively nonequivalent representations	196

7. Reduction to Smaller Groups	199
7.1. Some preliminary results	199
7.2. Inflated modules	203
7.3. Reduction to smaller groups	209
Part II The Schur Multiplier	213
8. Operator Groups and Bilinear Forms	215
8.1. Operator groups	215
8.2. Bilinear forms	222
9. Free Groups, Homology and Resolutions	231
9.1. An excursion into cohomology and homology theories	232
9.1.A. Some module isomorphisms	232
9.1.B. Review of some group homology and cohomology	236
9.1.C. Reduction theorems	244
9.1.D. The 5-term exact sequence	249
9.1.E. Finiteness of homology groups	251
9.2. Free groups	253
9.3. Cohomology and homology of free groups	257
9.4. Presentations of groups	258
9.5. Subgroups of free groups	260
9.6. The Gruenberg resolution	264
9.7. Hopf's formula	269
9.8. Hereditary rings	270
9.9. The universal coefficient theorem	271

10. Generalities	275
10.1. Basic properties	276
10.1.A. Definitions and some general results	276
10.1.B. Some bounds for the Schur multiplier	283
10.1.C. Metacyclic groups	288
10.2. The Schur multiplier of operator groups	293
10.3. Stability	298
10.4. Primary components	300
10.5. An introduction to local control	306
10.6. Direct products	311
10.7. Abelian groups	317
10.7.A. General properties	317
10.7.B. Covering groups of abelian groups	323
10.8. Elementary abelian Sylow subgroups	326
10.9. Triviality of primary components	330
10.10. Semidirect products	333
10.11. Exponents of Schur multipliers	337
10.12. Nilpotent groups	342
10.12.A. Preliminaries	342
10.12.B. Cocycles of nilpotent groups	345
10.12.C. Some applications	352
10.13. Dimension subgroups and Schur multiplier	355
10.13.A. Polynomial cocycles of degree ≤ 1	356
10.13.B. A reduction to finite p -groups	359
10.13.C. Polynomial maps	362
10.13.D. The third dimension subgroup	364
10.14. Subgroups of the Schur multiplier	366
11. Schur's Formula and Applications	375
11.1. Group-theoretic preliminaries	375
11.2. Schur's formula and elementary applications	378
11.3. Computational workshop	389
11.3.A. Introduction and methods	389
11.3.B. Calculations	395
11.4. Schur's formula and operator groups	410

11.5.	Some exact sequences	415
11.5.A.	Universal coefficient theorem	416
11.5.B.	Stem extensions and stem covers	420
11.5.C.	The 5-term homology sequence	424
11.5.D.	The Ganea map	427
11.5.E.	Applications to nilpotent groups	432
11.6.	Isoclinisms and covering groups	438
11.7.	Quasisimple and perfect groups	444
11.8.	Capable and unicentral groups	452
11.8.A.	Introduction	452
11.8.B.	Capable groups	454
11.8.C.	Unicentral groups	457
11.8.D.	The main result	461
11.8.E.	Extra-special p -groups	468
11.9.	Schur multiplier and commutator relations	469
11.10.	Cyclic extensions	482
11.10.A.	Computation of $H_2(G, \mathbb{Z})$ via bar resolution	482
11.10.B.	Cyclic extensions	484
11.11.	Deficiency, efficiency and Schur multipliers	490
11.11.A.	The Reidemeister-Schreier theorem	490
11.11.B.	The deficiency and efficiency of a finite group	492
11.11.C.	Swan's example	493
11.11.D.	Another example of zero deficiency	494
11.11.E.	Efficiency of direct powers of groups	499
11.12.	Covering groups of direct products	508
12.	Symmetric and Alternating Groups	515
12.1.	Presentations	515
12.2.	The Schur multiplier of S_n	517
12.3.	The Schur multiplier of A_n	523
13.	Schur Multipliers of p-Groups	533
13.1.	The order of $M(G)$	534
13.2.	Some inequalities	546
13.3.	Relations between $M(G)$ and $H^i(G, \mathbb{F}_p)$, $i = 1, 2$	556
13.4.	Some properties of $H^2(G, \mathbb{F}_p)$	558

13.5.	The Golod-Šafarevič theorem	565
13.6.	A necessary condition for triviality of $M(G)$	577
13.7.	An upper bound for $d(M(G))$	582
13.8.	The series $\lambda_n(G)$	584
13.9.	An accuracy of the upper bound	595
13.10.	Multipliers of p -groups of class 2	597
14.	Cohomological G-functors	609
14.1.	Definitions and examples	609
14.2.	General properties and applications	627
14.3.	Singularities	638
14.4.	The Yoshida transfer theorem	655
14.5.	A digression into group theory	658
14.6.	Local control of Schur multipliers	664
	14.6.A. Introduction	664
	14.6.B. Maximal subgroups	665
	14.6.C. Normal subgroups and singularities	668
	14.6.D. Sasaki's theorems	670
14.7.	A generalization of Swan's theorem	677
15.	Wreath Products	683
15.1.	Group-theoretic preliminaries	683
15.2.	Statements of results	691
15.3.	Proof of Theorem 2.1	703
16.	Linear Groups	719
16.1.	General information on linear groups	719
	16.1.A. Linear groups over local rings	719
	16.1.B. Linear groups over fields	724
	16.1.C. Defining relations for special linear groups	727
16.2.	Linear groups over finite fields	729
16.3.	Schur multipliers of linear groups	733
16.4.	An alternative approach to the case $n = 2$	752

17. Some Simple Groups	759
17.1. The Mathieu groups	759
17.2. The Ree groups	762
17.3. Janko's first group	765
17.4. The Suzuki groups	766
17.4.A. Lie algebras	766
17.4.B. A lemma of Lie type	770
17.4.C. The Schur multiplier of Suzuki groups	772
18. Schur Multipliers and Orthogonal Modules	777
18.1. Inner product spaces	777
18.2. Tensor algebras	784
18.3. Clifford algebras	787
18.4. Double covers	792
18.5. Orthogonal modules	799
18.6. Main results	807
19. Finite Coxeter Groups	815
19.1. Group-theoretic preliminaries	815
19.2. Projective representations of irreducible Coxeter groups	818
19.3. Construction of matrices	823
19.4. Main results	826
Bibliography	831
Notation	885
Index	893

This page intentionally left blank

Part I

Projective

Representations : I

Projective representation theory originated in a series of papers by Schur (1904, 1907, 1911) who brought it to a remarkably mature form. Whereas ordinary representations are homomorphisms into linear groups, projective representations are homomorphisms into projective linear groups. Schur was the first to discover that not only do projective representations arise naturally in group-theoretic problems, but, more important, their study leads to new theorems about ordinary representations. Subsequently, this phenomenon was also demonstrated by Clifford (1937) who studied connections between the representations of a group and those of its normal subgroups and factor groups.

The next important step was undertaken by Mackey (1958) who generalized much of the theory to separable locally compact groups. He vividly demonstrated that the projective representation theory is self-contained in a manner that the ordinary theory is not. An important contribution of Mackey was the extension to the infinite case of a classical device of Schur which enables one to deduce theorems about projective representations from corresponding ones about ordinary representations.

It turns out that in some cases minor modifications in the ordinary proof lead at once to a proof for projective representations. In other cases, either supplementary considerations are needed, or when a parallelism with ordinary representations disappears, a certain reformulation of the theorems is necessary.

For example, if $\alpha \in Z^2(G, \mathbb{C}^*)$, then Schur discovered that the number of irreducible projective representations of a finite group G over \mathbb{C} associated

with α is equal to the number of α -regular conjugacy classes of G , namely those classes for which $\alpha(x, g) = \alpha(g, x)$ whenever g is in the class and $gx = xg$. This example shows that, there are fewer irreducible projective representations than irreducible ordinary representations.

Another illuminating example is that a cyclic group of order 2 can have an irreducible projective representation over \mathbb{Q} of degree 2, which is clearly impossible for the ordinary representations. The reason for this phenomenon is that the field $\mathbb{Q}(\sqrt{2})$ is a twisted group algebra of \mathbb{Z}_2 over \mathbb{Q} .

As a final example, we mention that the degrees of irreducible projective representations of a finite group G over \mathbb{C} need not divide the index of an abelian normal subgroup of G , e.g. take $G = A_4$, the alternating group of degree 4.

In addition to the mentioned contributions of Schur, Mackey and Clifford, further results have been obtained by many researchers and the subject remains an attractive one to study. Our presentation of projective representation theory consists of two parts, both of which rely heavily on the modern treatment via modules over twisted group algebras. This gives a unified framework for all the material discussed above, as well as introducing some new machinery coming from ring theory. A systematic description of the material is supplied by detailed introductions to individual chapters and therefore will not be repeated here.

Chapter 1

Second Cohomology Groups

The study of projective representations of a group G over a field F breaks up into two parts. First, one attempts to determine the structure of the second cohomology group $H^2(G, F^*)$, where F^* is the multiplicative group of F . Then, for a given cohomology class, one tries to classify all projective representations of G associated with it. Our aim here is to concentrate exclusively on the first part. We do it in a more general context of arbitrary abelian coefficient groups A .

After reviewing basic facts arising from the general cohomology theory presented in Vol.1, we concentrate on cohomology groups of direct products and prove that if $G = G_1 \times G_2$, then $H^2(G, A)$ is the direct product of $H^2(G_1, A)$, $H^2(G_2, A)$ and $\text{Hom}(G_1 \otimes G_2, A)$. As an application, we then determine isomorphism classes of second cohomology groups of finite abelian groups.

Our next result reduces the study of $H^2(G, A)$ to that of $H^2(G, t(A))$. More precisely, we prove that if G is finite, then $H^2(G, A)$ is the direct product of $H^2(G, t(A))$ and $H^2(G, A/t(A))$, where $t(A)$ denotes the torsion subgroup of the abelian group A . We subsequently determine the isomorphism class of the second factor by exploiting the fact that $A/t(A)$ is torsion-free.

Special attention is drawn to an important case in which A is the multiplicative group of a field. A separate section provides a detailed analysis of this situation. The chapter ends with an investigation of cohomology of semidirect products of groups which act trivially on an abelian group A . It is demonstrated that in this case the cohomology groups with nontrivial action appear in an unavoidable way. Thus the cohomology theory with arbitrary actions is self-contained in a way that the corresponding theory with trivial

actions is not.

1 Review of basic facts

In what follows, G is an arbitrary group and A is an abelian multiplicative group. We also assume that G acts on A (equivalently, A is a $\mathbb{Z}G$ -module). Thus, for each $a \in A$ and $g \in G$, there is a unique element ${}^g a \in A$ such that

$$\begin{aligned} {}^1 a &= a && \text{for all } a \in A \\ {}^{xy} a &= {}^x ({}^y a) && \text{for all } x, y \in G, a \in A \\ {}^g (ab) &= {}^g a {}^g b && \text{for all } g \in G, a, b \in A \end{aligned}$$

Given a subgroup H of G , we write

$$A^H = \{a \in A \mid {}^h a = a \text{ for all } h \in H\}$$

Expressed otherwise, A^H is the largest subgroup of A on which H acts trivially. We now recall an explicit definition of the cohomology groups $H^1(G, A)$ and $H^2(G, A)$ with respect to the standard resolution of \mathbb{Z} .

A map $f : G \rightarrow A$ is called a **crossed homomorphism** if

$$f(xy) = {}^x f(y) f(x) \quad \text{for all } x, y \in G$$

If f and g are crossed homomorphisms, then their product fg defined by

$$(fg)(x) = f(x)g(x) \quad (x \in G)$$

is again a crossed homomorphism. It follows easily that the set $Z^1(G, A)$ of all crossed homomorphisms of G into A is an abelian group.

Given $a \in A$, the map $f_a : G \rightarrow A$ defined by

$$f_a(x) = {}^x a \cdot a^{-1}$$

is obviously a crossed homomorphism. We shall refer to f_a as a **principal crossed homomorphism** and denote by $B^1(G, A)$ the set of all such homomorphisms. It is clear that $B^1(G, A)$ is a subgroup of $Z^1(G, A)$. The corresponding factor group

$$H^1(G, A) = Z^1(G, A)/B^1(G, A)$$

is called the **first cohomology group** of G over A .

If G acts trivially on A , then the crossed homomorphisms become ordinary homomorphisms and each principal homomorphism $f : G \rightarrow A$ satisfies $f(g) = 1$ for all $g \in G$. Thus in this case

$$H^1(G, A) = \text{Hom}(G, A)$$

where $\text{Hom}(G, A)$ denotes the abelian group of all homomorphisms of G into the abelian group A .

We next recall an interpretation of $Z^1(G, A)$ and $H^1(G, A)$ in terms of splitting extensions of A by G . An **extension** of A by G is a short exact sequence of groups :

$$1 \rightarrow A \xrightarrow{\alpha} X \xrightarrow{\beta} G \rightarrow 1$$

We say that the above extension **splits** if there is a homomorphism

$$\gamma : G \rightarrow X$$

such that $\beta \circ \gamma = 1_G$. We refer to any such homomorphism γ as a **splitting homomorphism** .

Assume that $1 \rightarrow A \xrightarrow{i} X \xrightarrow{f} G \rightarrow 1$ (i is the inclusion map) is a splitting extension of A by G and let $\mu : G \rightarrow X$ be a splitting homomorphism. For any $a \in A$, the map

$${}^a\mu : G \rightarrow X$$

defined by

$${}^a\mu(g) = a\mu(g)a^{-1}$$

is also a splitting homomorphism. We shall refer to ${}^a\mu$ as being **A -conjugate** of μ . It is clear that A -conjugacy is an equivalence relation on the set of all splitting homomorphisms $G \rightarrow X$.

Theorem 1.1. *Let $1 \rightarrow A \xrightarrow{i} X \xrightarrow{f} G \rightarrow 1$ be a splitting extension of A by G , let $\mu : G \rightarrow X$ be a splitting homomorphism and let $H^1(G, A)$ be defined with respect to the following action of G on A :*

$${}^g a = \mu(g)a\mu(g)^{-1} \quad (a \in A, g \in G)$$

(i) *For each $\alpha \in Z^1(G, A)$, the map $\alpha\mu : G \rightarrow X$ defined by*

$$(\alpha\mu)(g) = \alpha(g)\mu(g) \quad (g \in G)$$