NORTH-HOLLAND MATHEMATICS STUDIES 177
(Continuation of the Notas de Matemática)

Editor: Leopoldo NACHBIN

Centro Brasileiro de Pesquisas Físicas
Rio de Janeiro, Brazil
and
University of Rochester
New York, U.S.A.

NORTH-HOLLAND – AMSTERDAM • LONDON • NEW YORK • TOKYO

GROUP REPRESENTATIONS
Volume 2

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1993

NORTH-HOLLAND – AMSTERDAM • LONDON • NEW YORK • TOKYO

ELSEVIER SCIENCE PUBLISHERS B.V.
Sara Burgerhartstraat 25
P.O. Box 211, 1000 AE Amsterdam, The Netherlands

ISBN: 0 444 88726 1
(Volume 1 (Part A and B): ISBN 0 444 88632 X)

This book is printed on acid-free paper.

Printed in The Netherlands

For Helen

who helped me most

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Preface

The present book is the second volume of a multi-volume treatise on group representations. The principal object of the volume is to provide, in a self-contained manner, a comprehensive coverage of projective representations and the Schur multiplier. Some further topics pertaining to projective representations will be covered in the next volume.

The book is a compilation and synthesis of a large body of work, done over several decades. It is aimed to give a clear and lucid exposition of the span of the field. The pace and style is appropriate for a graduate text, and could be used as an entry into the field as well as a basic reference. I hope that the volume will not only serve the specialist as a valuable reference, but will also provide ready access to the key ideas of the theory to the interested nonspecialist and to the student.

The bibliography is extensive, and it leads the reader to various references for detailed discussions on the main topics as well as on related subjects. A word about notation. As is customary, Theorem 4.3.2 denotes the second result in Section 3 of Chapter 4; however, for simplicity all references to this result within Chapter 4 itself, are designated as Theorem 3.2. The reader who wishes to obtain a more detailed summary account of the contents of this book can have it by reading through the brief introductions with which I begin each chapter.

I would like to express my gratitude to my wife for the tremendous help and encouragement she has given me in the preparation of this book. For answering specific queries on topics contained in the text I am indebted to R. Griess, D. Holt, J.-P. Serre, R. Steinberg, R. Swan, C. Thomas and T. Yoshida.

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November, 1992

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Part I
Projective Representations : I

Projective representation theory originated in a series of papers by Schur (1904, 1907, 1911) who brought it to a remarkably mature form. Whereas ordinary representations are homomorphisms into linear groups, projective representations are homomorphisms into projective linear groups. Schur was the first to discover that not only do projective representations arise naturally in group-theoretic problems, but, more important, their study leads to new theorems about ordinary representations. Subsequently, this phenomenon was also demonstrated by Clifford (1937) who studied connections between the representations of a group and those of its normal subgroups and factor groups.

The next important step was undertaken by Mackey (1958) who generalized much of the theory to separable locally compact groups. He vividly demonstrated that the projective representation theory is self-contained in a manner that the ordinary theory is not. An important contribution of Mackey was the extension to the infinite case of a classical device of Schur which enables one to deduce theorems about projective representations from corresponding ones about ordinary representations.

It turns out that in some cases minor modifications in the ordinary proof lead at once to a proof for projective representations. In other cases, either supplementary considerations are needed, or when a parallelism with ordinary representations disappears, a certain reformulation of the theorems is necessary.

For example, if $\alpha \in Z^2(G, \mathbb{C}^*)$, then Schur discovered that the number of irreducible projective representations of a finite group G over $\mathbb{C}$ associated with α is equal to the number of α-regular conjugacy classes of G, namely those classes for which $\alpha(x,g) = \alpha(g,x)$ whenever g is in the class and $gx = xg$. This example shows that, there are fewer irreducible projective representations than irreducible ordinary representations.

Another illuminating example is that a cyclic group of order 2 can have an irreducible projective representation over $\mathbb{Q}$ of degree 2, which is clearly impossible for the ordinary representations. The reason for this phenomenon is that the field $\mathbb{Q}(\sqrt{2})$ is a twisted group algebra of $\mathbb{Z}_2$ over $\mathbb{Q}$.

As a final example, we mention that the degrees of irreducible projective representations of a finite group G over $\mathbb{C}$ need not divide the index of an abelian normal subgroup of G, e.g. take $G = A_4$, the alternating group of degree 4.

In addition to the mentioned contributions of Schur, Mackey and Clifford, further results have been obtained by many researchers and the subject remains an attractive one to study. Our presentation of projective representation theory consists of two parts, both of which rely heavily on the modern treatment via modules over twisted group algebras. This gives a unified framework for all the material discussed above, as well as introducing some new machinery coming from ring theory. A systematic description of the material is supplied by detailed introductions to individual chapters and therefore will not be repeated here.

Chapter 1

Second Cohomology Groups

The study of projective representations of a group G over a field F breaks up into two parts. First, one attempts to determine the structure of the second cohomology group $H^2(G, F^*)$, where F^* is the multiplicative group of F. Then, for a given cohomology class, one tries to classify all projective representations of G associated with it. Our aim here is to concentrate exclusively on the first part. We do it in a more general context of arbitrary abelian coefficient groups A.

After reviewing basic facts arising from the general cohomology theory presented in Vol.1, we concentrate on cohomology groups of direct products and prove that if $G = G_1 \times G_2$, then $H^2(G, A)$ is the direct product of $H^2(G_1, A)$, $H^2(G_2, A)$ and $Hom(G_1 \otimes G_2, A)$. As an application, we then determine isomorphism classes of second cohomology groups of finite abelian groups.

Our next result reduces the study of $H^2(G, A)$ to that of $H^2(G, t(A))$. More precisely, we prove that if G is finite, then $H^2(G, A)$ is the direct product of $H^2(G, t(A))$ and $H^2(G, A/t(A))$, where $t(A)$ denotes the torsion subgroup of the abelian group A. We subsequently determine the isomorphism class of the second factor by exploiting the fact that $A/t(A)$ is torsion-free.

Special attention is drawn to an important case in which A is the multiplicative group of a field. A separate section provides a detailed analysis of this situation. The chapter ends with an investigation of cohomology of semidirect products of groups which act trivially on an abelian group A. It is demonstrated that in this case the cohomology groups with nontrivial action appear in an unavoidable way. Thus the cohomology theory with arbitrary actions is self-contained in a way that the corresponding theory with trivial actions is not.

1 Review of basic facts

In what follows, G is an arbitrary group and A is an abelian multiplicative group. We also assume that G acts on A (equivalently, A is a $\mathbb{Z}G$-module). Thus, for each $a \in A$ and $g \in G$, there is a unique element ${}^g a \in A$ such that

$$\begin{array}{rll} {}^1 a = a & \text{for all} & a \in A \\ {}^{xy} a = {}^x({}^y a) & \text{for all} & x, y \in G, a \in A \\ {}^g(ab) = {}^g a\, {}^g b & \text{for all} & g \in G, a, b \in A \end{array}$$

Given a subgroup H of G, we write

$$A^H = \{a \in A |\, {}^h a = a \quad \text{for all} \quad h \in H\}$$

Expressed otherwise, A^H is the largest subgroup of A on which H acts trivially. We now recall an explicit definition of the cohomology groups $H^1(G, A)$ and $H^2(G, A)$ with respect to the standard resolution of $\mathbb{Z}$.

A map $f : G \to A$ is called a **crossed homomorphism** if

$$f(xy) = {}^x f(y) f(x) \qquad \text{for all} \quad x, y \in G$$

If f and g are crossed homomorphisms, then their product fg defined by

$$(fg)(x) = f(x)g(x) \qquad (x \in G)$$

is again a crossed homomorphism. It follows easily that the set $Z^1(G, A)$ of all crossed homomorphisms of G into A is an abelian group.

Given $a \in A$, the map $f_a : G \to A$ defined by

$$f_a(x) = {}^x a \cdot a^{-1}$$

is obviously a crossed homomorphism. We shall refer to f_a as a **principal crossed homomorphism** and denote by $B^1(G, A)$ the set of all such homomorphisms. It is clear that $B^1(G, A)$ is a subgroup of $Z^1(G, A)$. The corresponding factor group

$$H^1(G, A) = Z^1(G, A)/B^1(G, A)$$

is called the **first cohomology group** of G over A.

If G acts trivially on A, then the crossed homomorphisms become ordinary homomorphisms and each principal homomorphism $f : G \to A$ satisfies $f(g) = 1$ for all $g \in G$. Thus in this case

$$H^1(G, A) = Hom(G, A)$$

where $Hom(G, A)$ denotes the abelian group of all homomorphisms of G into the abelian group A.

We next recall an interpretation of $Z^1(G, A)$ and $H^1(G, A)$ in terms of splitting extensions of A by G. An **extension** of A by G is a short exact sequence of groups :

$$1 \to A \xrightarrow{\alpha} X \xrightarrow{\beta} G \to 1$$

We say that the above extension **splits** if there is a homomorphism

$$\gamma : G \to X$$

such that $\beta \circ \gamma = 1_G$. We refer to any such homomorphism γ as a **splitting homomorphism** .

Assume that $1 \to A \xrightarrow{i} X \xrightarrow{f} G \to 1$ (i is the inclusion map) is a splitting extension of A by G and let $\mu : G \to X$ be a splitting homomorphism. For any $a \in A$, the map

$${}^a\mu : G \to X$$

defined by

$${}^a\mu(g) = a\mu(g)a^{-1}$$

is also a splitting homomorphism. We shall refer to ${}^a\mu$ as being A-**conjugate** of μ. It is clear that A-conjugacy is an equivalence relation on the set of all splitting homomorphisms $G \to X$.

Theorem 1.1. *Let $1 \to A \xrightarrow{i} X \xrightarrow{f} G \to 1$ be a splitting extension of A by G, let $\mu : G \to X$ be a splitting homomorphism and let $H^1(G, A)$ be defined with respect to the following action of G on A :*

$${}^g a = \mu(g)a\mu(g)^{-1} \qquad (a \in A, g \in G)$$

(i) For each $\alpha \in Z^1(G, A)$, the map $\alpha\mu : G \to X$ defined by

$$(\alpha\mu)(g) = \alpha(g)\mu(g) \qquad (g \in G)$$